Exercises, Summer school on the Hitchin fibration

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Exercise 1. Let $z = x_1 + \mathrm{i}x_2$ be the standard holomorphic coordinate on an open set $U \subset \mathbf{C}$, and set

$$\nabla = \mathrm{d} + A_1 \mathrm{d}x_1 + A_2 \mathrm{d}x_2$$

and

$$\Phi = \frac{1}{2}(\phi_1 - \mathrm{i}\phi_2)\mathrm{d}z \in \Omega^{1,0}(C, \mathrm{ad}(P)^{\mathbf{C}}).$$

Show that the Self-Duality Equations are equivalent to

$$F_\nabla + [\Phi, \Phi^*] = 0 \qquad (\mathbf{R}\mathrm{SD})$$

$$\nabla^{0,1}\Phi = 0, \qquad (\mathbf{C}\mathrm{SD})$$

where ∇ acts on endomorphisms by the adjoint action.

Exercise 2. Set $\mathcal{G} = \Omega^0(C, \mathrm{Ad}(P))$ and $\mathcal{G}^{\mathbf{C}} = \Omega^0(C, \mathrm{Ad}(P)^{\mathbf{C}})$. Show that for any $g \in \mathcal{G}^{\mathbf{C}}$, if (∇, Φ) is a solution of ($\mathbf{C}$SD) then so is $g \cdot (\nabla, \Phi)$. Show that for any $g \in \mathcal{G}$, if (∇, Φ) is moreover a solution of ($\mathbf{R}$SD) then so is $g \cdot (\nabla, \Phi)$.

Exercise 3. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional Hermitian vector space and G a finite dimensional compact connected Lie group. Recall that V carries a natural Kähler-structure defined by the Riemannian metric

$$g(\mathbf{v}, \mathbf{w}) = \Re\langle \mathbf{v}, \mathbf{w} \rangle$$

and the symplectic form

$$\omega(\mathbf{v}, \mathbf{w}) = g(\mathrm{i}\mathbf{v}, \mathbf{w}) = \Im\langle \mathbf{v}, \mathbf{w} \rangle$$

for any $\mathbf{v}, \mathbf{w} \in V$. Assume the compactification $G^{\mathbf{C}}$ of G acts on V smoothly by linear transformations, and assume that the action restricted to G preserves the Hermitian structure too. Define the map $\mu : V \to \mathfrak{g}^\vee$ by

$$\mu(\mathbf{v})(\xi) = \frac{\mathrm{i}}{2}\langle \xi \cdot \mathbf{v}, \mathbf{v} \rangle,$$

for any $\xi \in \mathfrak{g}$ and $\mathbf{v} \in V$, where $\xi \cdot \mathbf{v}$ stands for the infinitesimal action of $\mathfrak{g}$ on V. Show that μ is a moment map of the restricted action of G.

Exercise 4. With the notations of Exercise 3, let now $\chi : G \to \mathrm{U}(1)$ be any character of G, and call the induced complex character $\chi^{\mathbf{C}} : G^{\mathbf{C}} \to \mathbf{C}^*$. Denote by $\mathrm{d}\chi : \mathfrak{g} \to \mathrm{i}\mathbf{R}$ the derivative of χ; in other words, we have $\mathrm{i}(\mathrm{d}\chi) \in \mathfrak{g}^\vee$. The action of $G^{\mathbf{C}}$ on V and the character $\chi^{\mathbf{C}}$ give rise to the action of $G^{\mathbf{C}}$ on $V \times \mathbf{C}$ defined by the formula

$$g \cdot (\mathbf{v}, z) = (g \cdot \mathbf{v}, \chi^{\mathbf{C}}(g)^{-1} z).$$

We say that $\mathbf{v} \in V$ is χ-stable if for all $z \in \mathbf{C}^*$ the orbit of $(\mathbf{v}, z)$ under this action is closed in $V \times \mathbf{C}$; denote by $V^{\chi-s}$ the set of χ-stable points $\mathbf{v}$ of V. A generalisation of the Theorem seen in class states that there exists a one-to-one correspondence between $\mu^{-1}(\mathrm{i}(\mathrm{d}\chi))/G$ and $V^{\chi-s}/G^{\mathbf{C}}$.

1. Apply this result to the standard action of $\mathrm{U}(1)$ on $\mathbf{C}^{n+1}$ and the character $\chi_l : \mathrm{U}(1) \to \mathrm{U}(1)$ given by $g \mapsto g^l$ for a fixed $l \in \mathbf{Z}$. What are the χ_l-stable points, what is the moment map, and what are the quotients on the two sides of the correspondence?

2. Let $0 < k < n$ be integers. Apply the above result to the case where $V = \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^k, \mathbf{C}^n)$, $G = \mathrm{U}(k)$ acting by $g \cdot \varphi = \varphi \circ g^{-1}$ for any $\varphi \in V$ and $g \in G$, and $\chi = \det^{-1} : G \to \mathrm{U}(1)$.

Exercise 5. Let $n \in \mathbf{N}$, $V = \mathrm{End}(\mathbf{C}^n)$ and $G = \mathrm{U}(n)$ act on it by conjugation:

$$g \cdot \varphi = g \circ \varphi \circ g^{-1}.$$

Find the moment map μ of this action, the set of stable points of V and compare the two quotients.

Exercise 6. (Atiyah-Bott) Let $G = \mathrm{U}(n)$, P be a G-bundle on C and $\mathcal{A}$ denote the affine space of unitary connections ∇ on P. Recall that ∇ is uniquely determined by its $(0,1)$-part $\nabla^{0,1}$, therefore the tangent space to $\mathcal{A}$ at ∇ can be identified to $\Omega^{0,1}(C, \mathrm{ad}(P)^{\mathbf{C}})$. Let the Kähler structure on $\mathcal{A} \times \Omega^{1,0}(C, \mathrm{ad}(P)^{\mathbf{C}})$ be given by the complex structure $(\psi, \phi) \mapsto (\mathrm{i}\psi, \mathrm{i}\phi)$ and the Hermitian metric

$$\langle (\psi_1, \phi_1), (\psi_2, \phi_2) \rangle = 2\mathrm{i} \int_C \mathrm{tr}(\psi_2^* \wedge \psi_1 + \phi_1 \wedge \phi_2^*).$$

Moreover, let $\mathcal{G} = \Omega^0(C, \mathrm{Ad}(P))$ act on $\mathcal{A} \times \Omega^{1,0}(C, \mathrm{ad}(P)^{\mathbf{C}})$ by conjugation. Observe that the Lie-algebra $\mathrm{Lie}(\mathcal{G})$ of $\mathcal{G}$ can be naturally identified with $\Omega^0(C, \mathrm{ad}(P))$, and its dual with $\Omega^2(C, \mathrm{ad}(P))$ by the Killing form and integration over C. Show that the moment map of the action of $\mathcal{G}$ on $\mathcal{A} \times \Omega^{1,0}(C, \mathrm{ad}(P)^{\mathbf{C}})$ is then

$$\begin{aligned} \mu : \mathcal{A} \times \Omega^{1,0}(C, \mathrm{ad}(P)^{\mathbf{C}}) &\to \Omega^2(C, \mathrm{ad}(P)) \\ (\nabla, \Phi) &\mapsto F_\nabla + [\Phi, \Phi^*]. \end{aligned}$$

Exercise 7. Consider the 4-th power $(\mathbf{CP}^1)^4$ of $\mathbf{CP}^1$, and let Δ be the diagonal composed of quadruples in which not all points are distinct. Let the group $G^{\mathbf{C}} = \mathrm{Sl}_2(\mathbf{C})$ act on $X = (\mathbf{CP}^1)^4 \setminus \Delta$ diagonally by the standard action. Determine the set of stable points, the moment map of the action restricted to $G = \mathrm{U}(2)$ and the various quotient spaces.