Exercises for the Course on Perverse Sheaves

Summer School on the Hitchin system
Bonn, July 2010

Exercise 1

(a) Let $\mathcal{T}$ be a triangulated category and $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ a (distinguished) triangle. If u is a monomorphism then this triangle is isomorphic to the triangle $X \to X \oplus Z \to Z \xrightarrow{0} X[1]$ with the obvious morphisms.

(b) **Splitting criterion:** Let $\mathcal{T}$ be the derived category of an abelian category with its standard t-structure (or, more generally, a triangulated category with a non degenerate t-structure).

Let $C \xrightarrow{u} A \xrightarrow{v} B \xrightarrow{w} [1]C$ be a (distinguished) triangle with $A \in \mathcal{T}^{\leq 0}$ and $C \in \mathcal{T}^{\geq 0}$. If $H^0(u) : H^0(C) \to H^0(A)$ is an isomorphism then $A \cong H^0(C) \oplus \tau_{\leq -1} B$.

Exercise 2

(a) Let C be a complex curve. Then the constant sheaf $\mathbb{Q}_C[1]$ in degree -1 is perverse. When is it simple?

(b) Let S be a complex surface with isolated singularities. When is $\mathbb{Q}_S[2]$ perverse? When is it simple?

Exercise 3

Deduce the following statements from the Decomposition Theorem:

(a) If $f : X \to Y$ is a resolution of singularities, then $\mathcal{IC}_Y$ appears in $f_*\mathcal{IC}_X = f_*\mathbb{Q}_X[\dim X]$ as a direct summand.

(b) Let $f : X \to Y$ be a proper smooth map of irreducible smooth varieties. (By a result of Ehresmann, f is a C^∞ fiber bundle.) Then

$$f_*\mathbb{Q}_X \cong \bigoplus_{i \in \mathbb{Z}} (H^i f_* \mathbb{Q}_X)[-i]$$

and all $H^i f_* \mathbb{Q}_X$ are semisimple local systems. Deduce that the Leray-Serre spectral sequence

$$E_2^{p,q} = H^p(Y; H^q f_* \mathbb{Q}_X) \Rightarrow H^{p+q}(X)$$

degenerates at the E_2-term.