Problems for Hausel's Lectures

1. Assume E and F are stable bundles on a smooth projective curve C of the same slope.

   (a) Show that if $f : E \to F$ is a non-zero homomorphism then it is an isomorphism.

   (b) Deduce that a stable bundle E is simple, i.e. any endomorphism of E is of the form λId_E.

   (c) As a consequence calculate $\dim T_{[E]}\mathcal{N} = \dim(H^1(C; \mathrm{End}(E))) = n^2(g-1)+1$ for a stable vector bundle E using Riemann-Roch.

   (d) Calculate also the dimension of the GL_n Hitchin base $\bigoplus_{i=1}^n H^0(C; K^i)$!

2. Let M be a non-singular complex algebraic variety. For $x = (m, \alpha) \in T^*M$, where $\alpha \in T^*_m M$ define $\theta_x : T_x T^*M \to \mathbb{C}$ given by $\theta_x(X) := \langle T_\pi(X), \alpha\rangle$, where $X \in T_x T^*M$ and $\pi : T^*M \to M$ is the projection. This defines a 1-form $\theta \in \Omega^1(T^*M)$ the so-called Liouville 1-form. Show that $\omega := d\theta \in \Omega^2(T^*M)$ is a nowhere degenerate 2-form, and because clearly $d\omega = 0$ it is a canonical symplectic form on the cotangent bundle of any variety M. When $M = \mathbb{C}^n$ find suitable coordinates on $T^*\mathbb{C}^n$ to get $\theta = \sum_i^n x_i dy_i$ and consequently $\omega = \sum_i^n dx_i \wedge dy_i$.

3. Determine the dimension of the affinization $\mathrm{Spec}\left(H^0(T^*M, \mathcal{O}_{T^*M})\right)$ when (a) M is a smooth projective curve of genus g (b) $M = \mathbb{P}^n$ or more generally a partial flag variety (c) M is a smooth projective toric variety.

4. Let E be a complex elliptic curve; N a degree 0 non-torsion invertible sheaf on E and P a degree -1 invertible sheaf on E; and let X be the total space of the rank 2 vector bundle $N \oplus P$ on E. Show that the algebra of global functions $H^0(X, \mathcal{O}_X)$ on X is not finitely generated. (Example due to Ravi Vakil.) (Hint: Consider the natural $(\mathbb{C}^\times)^2$ action on $N \oplus P$ and deduce that
   $H^0(X, \mathcal{O}_X) = \bigoplus_{k,l=0}^\infty H^0(E, N^{-k} \otimes P^{-l})$.)

5. Determine the moduli space of rank 1 stable Higgs bundles on a smooth projective curve C. Describe the Hitchin map explicitly.

6. Consider the space V_n of monic degree n complex polynomials and show that it is isomorphic with the space of unordered complex n-tuples. Deduce Chevalley's isomorphism $V_n = \mathbb{C}^n \cong \mathbb{C}^n/S^n$, where the symmetric group S^n acts on $\mathbb{C}^n$ by permuting the coordinates. Identify on $V_n = \mathbb{C}_n$ the $\mathbb{C}^\times$-action induced from multiplying unordered n-tuples with the same scalar. (The only natural structure on the space V_n is this $\mathbb{C}^\times$-action - the vector space structure is an accident.)